

Problem 2.21

[Difficulty: 3]

2.21 Consider the flow field given in Eulerian description by the expression $\vec{V} = A\hat{i} - Bt\hat{j}$, where $A = 2$ m/s, $B = 2$ m/s², and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0, 1$, and 2 s.

Given: Eulerian Velocity field

Find: Lagrangian position function that was at point (1,1) at $t = 0$; expression for pathline; plot pathline and compare to streamlines through same point at the instants $t = 0, 1$ and 2 s

Solution:

Governing equations: For pathlines (Lagrangian description) $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = A$ $A = 2 \frac{m}{s}$ $v_p = \frac{dy}{dt} = -B \cdot t$ $B = 2 \frac{m}{s^2}$

So, separating variables $dx = A \cdot dt$ $dy = -B \cdot t \cdot dt$

Integrating $x = A \cdot t + x_0$ $x_0 = 1 \text{ m}$ $y = -B \cdot \frac{t^2}{2} + y_0$ $y_0 = 1 \text{ m}$

The Lagrangian description is $x(t) = A \cdot t + x_0$ $y(t) = -B \cdot \frac{t^2}{2} + y_0$

Using given data $x(t) = 2 \cdot t + 1$ $y(t) = 1 - t^2$

The pathlines are given by combining the equations $t = \frac{x - x_0}{A}$ $y = -B \cdot \frac{t^2}{2} + y_0 = -B \cdot \frac{(x - x_0)^2}{2 \cdot A^2} + y_0$

Hence $y(x) = y_0 - B \cdot \frac{(x - x_0)^2}{2 \cdot A^2}$ or, using given data $y(x) = 1 - \frac{(x - 1)^2}{4}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{-B \cdot t}{A}$

So, separating variables $dy = -\frac{B \cdot t}{A} \cdot dx$ which we can integrate for any given t (t is treated as a constant)

The solution is

$$y = -\frac{B \cdot t}{A} \cdot x + c$$

and for the one through (1,1)

$$1 = -\frac{B \cdot t}{A} \cdot 1 + c$$

$$c = 1 + \frac{B \cdot t}{A}$$

$$y = -\frac{B \cdot t}{A} \cdot (x - 1) + 1$$

$$y = 1 - t \cdot (x - 1)$$

$$x = 1, 1.1 \dots 20$$

